Free-Response Scoring Guidelines

The following contains the scoring guidelines
for the free-response questions in this exam.

Question 1

t (hours)	0	0.4	0.8	1.2	1.6	2.0	2.4
$v(t)$ (miles per hour)	0	11.8	9.5	17.2	16.3	16.8	20.1

Ruth rode her bicycle on a straight trail. She recorded her velocity $v(t)$, in miles per hour, for selected values of t over the interval $0 \le t \le 2.4$ hours, as shown in the table above. For $0 < t \le 2.4$, $v(t) > 0$.

(a) Use the data in the table to approximate Ruth's acceleration at time $t = 1.4$ hours. Show the computations that lead to your answer. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{2.4} v(t)\,dt$ in the context of the problem. Approximate $\int_0^{2.4} v(t)\,dt$ using a midpoint Riemann sum with three subintervals of equal length and values from the table.

(c) For $0 \le t \le 2.4$ hours, Ruth's velocity can be modeled by the function g given by $g(t) = \dfrac{24t + 5\sin(6t)}{t + 0.7}$. According to the model, what was Ruth's average velocity during the time interval $0 \le t \le 2.4$?

(d) According to the model given in part (c), is Ruth's speed increasing or decreasing at time $t = 1.3$? Give a reason for your answer.

(a) $a(1.4) \approx \dfrac{v(1.6) - v(1.2)}{1.6 - 1.2} = \dfrac{16.3 - 17.2}{1.6 - 1.2} = -2.25 \text{ miles/hr}^2$

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{units} \end{cases}$

(b) $\int_0^{2.4} v(t)\,dt$ is the total distance Ruth traveled, in miles, from time $t = 0$ to time $t = 2.4$ hours.

$$\int_0^{2.4} v(t)\,dt \approx (0.8)(11.8) + (0.8)(17.2) + (0.8)(16.8) = 36.64 \text{ miles}$$

$3 : \begin{cases} 1 : \text{interpretation} \\ 1 : \text{midpoint Riemann sum} \\ 1 : \text{approximation} \end{cases}$

(c) Average velocity $= \dfrac{1}{2.4}\int_0^{2.4} g(t)\,dt = 14.064$ miles/hr

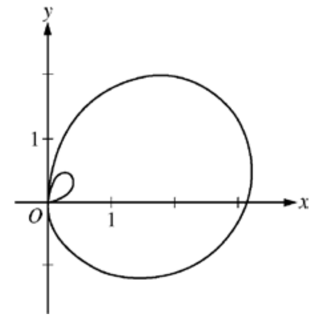
$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) Velocity $= g(1.3) = 18.096358 > 0$
Acceleration $= g'(1.3) = 3.761152 > 0$

Ruth's speed is increasing at time $t = 1.3$ since velocity and acceleration have the same sign at this time.

2 : conclusion with reason

Question 2



Consider the polar curve defined by the function $r(\theta) = \theta \cos \theta$, where $0 \le \theta \le \frac{3\pi}{2}$. The derivative of r is given by $\frac{dr}{d\theta} = \cos\theta - \theta\sin\theta$. The figure above shows the graph of r for $0 \le \theta \le \frac{3\pi}{2}$.

(a) Find the area of the region enclosed by the inner loop of the curve.

(b) For $0 \le \theta \le \frac{3\pi}{2}$, find the greatest distance from any point on the graph of r to the origin. Justify your answer.

(c) There is a point on the curve at which the slope of the line tangent to the curve is $\frac{2}{2-\pi}$. At this point, $\frac{dy}{d\theta} = \frac{1}{2}$. Find $\frac{dx}{d\theta}$ at this point.

(a) Area $= \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2 \, d\theta = 0.127$ (or 0.126)

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$

(b) $r'(\theta) = \cos\theta - \theta\sin\theta = 0 \Rightarrow \theta = 0.860334,\ 3.425618$

θ	$r(\theta)$
0	0
0.860334	0.561096
3.425618	−3.288371
$\frac{3\pi}{2}$	0

Therefore, the greatest distance from any point on the graph of r to the origin is 3.288.

$3 : \begin{cases} 1 : \text{identifies } \theta = 3.425618 \text{ as a candidate} \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(c) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

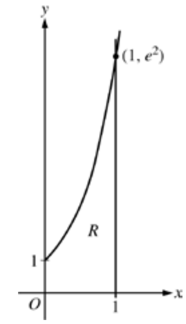
At the point where the tangent line has slope $\frac{2}{2-\pi}$,

$\frac{2}{2-\pi} = \frac{1/2}{dx/d\theta}$.

Therefore, $\frac{dx}{d\theta} = \frac{1}{2} \cdot \frac{2-\pi}{2} = \frac{2-\pi}{4}$ at this point.

$3 : \begin{cases} 1 : \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \\ 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$

Question 3



Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of $y = f(x)$ and the vertical line $x = 1$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$.

(b) Find the area of R.

(c) Region R forms the base of a solid whose cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(a) $f(1) = e^2$

$f'(x) = 2e^{2x} \Rightarrow f'(1) = 2e^2$

An equation for the tangent line is $y = e^2 + 2e^2(x - 1)$.

$$2 : \begin{cases} 1 : f'(1) \\ 1 : \text{answer} \end{cases}$$

(b) Area $= \int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_{x=0}^{x=1} = \frac{1}{2}\left(e^2 - 1\right)$

$$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

(c) Volume $= 1 + \int_1^{e^2} \left(1 - \frac{1}{2}\ln y\right)^2 dy$

$$4 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$$

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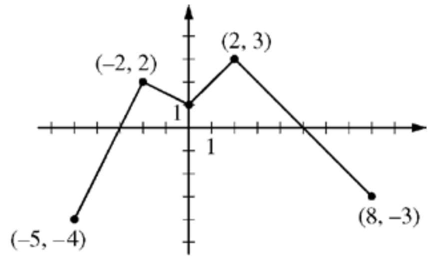
Question 4



Graph of f

The continuous function f is defined on the interval $-5 \le x \le 8$. The graph of f, which consists of four line segments, is shown in the figure above.

Let g be the function given by $g(x) = 2x + \int_{-2}^{x} f(t)\,dt$.

(a) Find $g(0)$ and $g(-5)$.

(b) Find $g'(x)$ in terms of $f(x)$. For each of $g''(4)$ and $g''(-2)$, find the value or state that it does not exist.

(c) On what intervals, if any, is the graph of g concave down? Give a reason for your answer.

(d) The function h is given by $h(x) = g\left(x^3 + 1\right)$. Find $h'(1)$. Show the work that leads to your answer.

(a) $g(0) = 2 \cdot 0 + \int_{-2}^{0} f(t)\,dt = 3$ $g(-5) = 2 \cdot (-5) + \int_{-2}^{-5} f(t)\,dt = -10 + 3 = -7$	$2 : \begin{cases} 1 : g(0) \\ 1 : g(-5) \end{cases}$
(b) $g'(x) = 2 + f(x)$ $g''(x) = f'(x)$ $g''(4) = f'(4) = -1$ $g''(-2) = f'(-2)$ does not exist.	$3 : \begin{cases} 1 : g'(x) \\ 1 : g''(4) \\ 1 : g''(-2) \text{ does not exist} \end{cases}$
(c) The graph of g is concave down on the intervals $(-2, 0)$ and $(2, 8)$ since $g'(x) = 2 + f(x)$ decreases on those intervals.	1 : intervals and reason
(d) $h'(x) = g'\left(x^3 + 1\right) \cdot 3x^2$ $h'(1) = g'(2) \cdot 3 = (2 + f(2)) \cdot 3$ $= (2 + 3) \cdot 3 = 15$	$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

Question 5

A toy train moves along a straight track set up on a table. The position $x(t)$ of the train at time t seconds is measured in centimeters from the center of the track. At time $t = 1,$ the train is 6 centimeters to the left of the center, so $x(1) = -6.$ For $0 \le t \le 4,$ the velocity of the train at time t is given by $v(t) = 3t^2 - 12,$ where $v(t)$ is measured in centimeters per second.

(a) For $0 \le t \le 4,$ find $x(t).$

(b) Find the total distance traveled by the train during the time interval $0 \le t \le 4.$

(c) A toy bus moving on the same table has position given by $(x(t), y(t)).$ Here, $x(t)$ is the function found in part (a), and $y(t) = 2 + 12\sin\left(\frac{t}{4}\right)$ is the distance from the bus to the train track, in centimeters. Write, but do not evaluate, an integral expression that gives the total distance traveled by the bus during the time interval $0 \le t \le 4.$

(a) $x(t) = -6 + \int_1^t \left(3u^2 - 12\right) du$

$= -6 + \left[u^3 - 12u\right]_{u=1}^{u=t}$

$= t^3 - 12t + 5$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) $v(t) = 3\left(t^2 - 4\right) = 0 \Rightarrow t = -2,\ t = 2$

Distance $= |x(2) - x(0)| + |x(4) - x(2)|$

$= |-11 - 5| + |21 - (-11)|$

$= 16 + 32 = 48$

$3 : \begin{cases} 1 : \text{identifies } t = 2 \\ 1 : \text{considers } x(0),\ x(2), \text{ and } x(4) \\ 1 : \text{answer} \end{cases}$

(c) $y'(t) = 3\cos\left(\frac{t}{4}\right)$

Distance $= \int_0^4 \sqrt{\left(3t^2 - 12\right)^2 + \left(3\cos\left(\frac{t}{4}\right)\right)^2}\, dt$

$3 : \begin{cases} 1 : y'(t) \\ 2 : \text{integral} \end{cases}$

Question 6

Let $a_n = \dfrac{1}{n \ln n}$ for $n \ge 3$.

(a) Let f be the function given by $f(x) = \dfrac{1}{x \ln x}$. For $x \ge 3$, f is continuous, decreasing, and positive.

Use the integral test to show that $\displaystyle\sum_{n=3}^{\infty} a_n$ diverges.

(b) Consider the infinite series $\displaystyle\sum_{n=3}^{\infty} (-1)^{n+1} a_n = \frac{1}{3\ln 3} - \frac{1}{4\ln 4} + \frac{1}{5\ln 5} - \cdots$. Identify properties of this series that guarantee the series converges. Explain why the sum of this series is less than $\dfrac{1}{3}$.

(c) Find the interval of convergence of the power series $\displaystyle\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$. Show the analysis that leads to your answer.

(a) $u = \ln x \Rightarrow du = \dfrac{1}{x}\,dx$

$$\int_3^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_{\ln 3}^{\ln b}\frac{1}{u}\,du = \lim_{b\to\infty}\Big[\ln|u|\Big]_{u=\ln 3}^{u=\ln b}$$
$$= \lim_{b\to\infty}\left(\ln|\ln b| - \ln|\ln 3|\right) = \infty$$

Therefore, the series diverges.

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{evaluation} \end{cases}$

(b) The terms in this alternating series decrease in absolute value and $\displaystyle\lim_{n\to\infty}\frac{1}{n\ln n} = 0$. Therefore, the Alternating Series Test guarantees that this series converges. Furthermore,

$$\frac{1}{3\ln 3} - \frac{1}{4\ln 4} < \text{Sum} < \frac{1}{3\ln 3} < \frac{1}{3}$$

Therefore, the sum of the series is less than $\dfrac{1}{3}$.

$2 : \begin{cases} 1 : \text{properties} \\ 1 : \text{explanation} \end{cases}$

(c) $$\left|\frac{(x-2)^{n+2}}{(n+1)\ln(n+1)} \cdot \frac{n\ln n}{(x-2)^{n+1}}\right| = \left(\frac{n}{n+1}\right)\left(\frac{\ln n}{\ln(n+1)}\right)|x-2|$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\frac{\ln n}{\ln(n+1)}\right)|x-2| = |x-2|$$

$|x-2| < 1 \Rightarrow 1 < x < 3$

When $x = 1$, the series is $\dfrac{1}{3\ln 3} - \dfrac{1}{4\ln 4} + \dfrac{1}{5\ln 5} - \dfrac{1}{6\ln 6} + \cdots$

This series converges by the Alternating Series Test.

When $x = 3$, the series is $\dfrac{1}{3\ln 3} + \dfrac{1}{4\ln 4} + \dfrac{1}{5\ln 5} + \dfrac{1}{6\ln 6} + \cdots$

This series diverges by the integral test, as shown in part (a).
Therefore, the interval of convergence is $1 \le x < 3$.

$4 : \begin{cases} 1\text{: sets up ratio} \\ 1\text{: computes limit of ratio} \\ 1\text{: identifies radius of convergence} \\ 1\text{: analysis and interval of convergence} \end{cases}$

Scoring Worksheets

The following provides a scoring worksheets and conversion tables used for calculating a composite score of the exam.