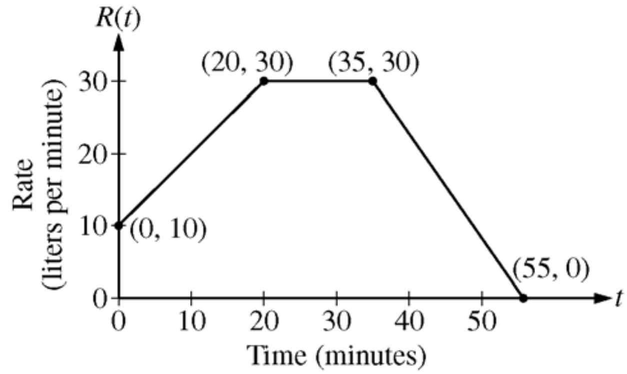
Free-Response Scoring Guidelines

The following contains the scoring guidelines
for the free-response questions in this exam.

AP® CALCULUS BC
2015 SCORING GUIDELINES

Question 1



At time $t = 0$ minutes, a tank contains 100 liters of water. The piecewise-linear graph above shows the rate $R(t)$, in liters per minute, at which water is pumped into the tank during a 55-minute period.

(a) Find $R'(45)$. Using appropriate units, explain the meaning of your answer in the context of this problem.

(b) How many liters of water have been pumped into the tank from time $t = 0$ to time $t = 55$ minutes? Show the work that leads to your answer.

(c) At time $t = 10$ minutes, water begins draining from the tank at a rate modeled by the function D, where $D(t) = 10e^{(\sin t)/10}$ liters per minute. Water continues to drain at this rate until time $t = 55$ minutes. How many liters of water are in the tank at time $t = 55$ minutes?

(d) Using the functions R and D, determine whether the amount of water in the tank is increasing or decreasing at time $t = 45$ minutes. Justify your answer.

(a) $R'(45) = \dfrac{30 - 0}{35 - 55} = -\dfrac{3}{2}$

The rate at which water is being pumped into the tank is decreasing at $\dfrac{3}{2}$ liters/min^2 at $t = 45$ minutes.

$2 : \begin{cases} 1 : R'(45) \\ 1 : \text{explanation} \end{cases}$

(b) $\displaystyle\int_0^{55} R(t)\,dt = 20 \cdot \frac{10 + 30}{2} + 15 \cdot 30 + \frac{1}{2} \cdot 20 \cdot 30$
$= 400 + 450 + 300 = 1150$

$2 : \begin{cases} 1 : \text{sum of areas} \\ 1 : \text{answer} \end{cases}$

(c) $\text{Amt} = 100 + 1150 - \displaystyle\int_{10}^{55} 10e^{(\sin t)/10}\,dt$
$= 1250 - 450.275371 = 799.725$ (or 799.724)

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{expression for water in the tank} \\ 1 : \text{answer} \end{cases}$

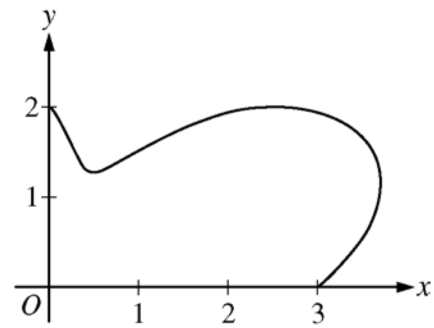
(d) $R(45) = 15$
$D(45) = 10.88815$

At time $t = 45$ minutes, the rate of water pumped into the tank is greater than the rate of water draining from the tank. Therefore, the amount of water in the tank is increasing at time $t = 45$ minutes.

2 : answer with justification

Question 2

The figure above shows the graph of the polar equation $r = 2 + \sin(4\theta) + \cos(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. The derivative of r with respect to θ is given by $r'(\theta) = 4\cos(4\theta) - \sin(\theta)$.



(a) Find the area of the region bounded by the graph of r and the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$.

(b) Find the area of the region in the first quadrant that is outside the graph of $r = 2 + \sin(4\theta) + \cos(\theta)$ but inside the graph of the circle of radius 2 centered at the origin.

(c) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve $r = 2 + \sin(4\theta) + \cos(\theta)$ with greatest distance from the origin. Justify your answer.

(a) Area $= \frac{1}{2}\int_0^{\pi/2} (2 + \sin(4\theta) + \cos(\theta))^2 \, d\theta$
$= 6.194$ (or 6.193)

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $2 + \sin(4\theta) + \cos(\theta) = 2 \Rightarrow \theta = 0.942478$
Let $c = 0.942478$

Area $= \frac{1}{2}\int_c^{\pi/2} \left[2^2 - (2 + \sin(4\theta) + \cos(\theta))^2\right] d\theta$
$= 0.456$

$4 : \begin{cases} 1 : \text{value of } \theta \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $r'(\theta) = 4\cos(4\theta) - \sin(\theta) = 0$
$\Rightarrow \theta = 0.370064, 1.237726$

θ	$r(\theta)$
0	3
0.370064	3.928208
1.237726	1.355256
$\pi/2$	2

$\theta = 0.370$ corresponds to the point on the curve with the greatest distance from the origin.

$3 : \begin{cases} 1 : \text{identifies } \theta = 0.370 \text{ as a candidate} \\ 2 : \text{answer with justification} \end{cases}$

Question 3

t (seconds)	0	3	5	8	12
$k(t)$ (feet per second)	0	5	10	20	24

Kathleen skates on a straight track. She starts from rest at the starting line at time $t = 0.$ For $0 < t \leq 12$ seconds, Kathleen's velocity $k,$ measured in feet per second, is differentiable and increasing. Values of $k(t)$ at various times t are given in the table above.

(a) Use the data in the table to estimate Kathleen's acceleration at time $t = 4$ seconds. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a right Riemann sum with the four subintervals indicated by the data in the table to approximate $\int_0^{12} k(t)\,dt.$ Indicate units of measure. Is this approximation an overestimate or an underestimate for the value of $\int_0^{12} k(t)\,dt$? Explain your reasoning.

(c) Nathan skates on the same track, starting 5 feet ahead of Kathleen at time $t = 0.$ Nathan's velocity, in feet per second, is given by $n(t) = \dfrac{150}{t+3} - 50e^{-t}.$ Write, but do not evaluate, an expression involving an integral that gives Nathan's distance from the starting line at time $t = 12$ seconds.

(d) Write an expression for Nathan's acceleration in terms of $t.$

(a) $a(4) \approx \dfrac{10-5}{5-3} = \dfrac{5}{2}$ ft/sec^2

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) $\int_0^{12} k(t)\,dt \approx (5)(3) + (10)(2) + (20)(3) + (24)(4) = 191$ feet

This approximation is an overestimate since a right Riemann sum is used and the function k is increasing.

$3 : \begin{cases} 1 : \text{right Riemann sum} \\ 1 : \text{approximation with units} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $s(12) = 5 + \int_0^{12} n(t)\,dt$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $n'(t) = (150)(-1)(t+3)^{-2} - 50e^{-t}(-1)$

$$= -\frac{150}{(t+3)^2} + 50e^{-t}$$

$2 : n'(t)$

Question 4

In a national park, the population of mountain lions grows over time. At time $t = 0$, where t is measured in years, the population is found to be 20 mountain lions.

(a) One zoologist suggests a population model P that satisfies the differential equation $\frac{dP}{dt} = \frac{1}{4}(220 - P)$. Use separation of variables to solve this differential equation for P with the initial condition $P(0) = 20$.

(b) A second zoologist suggests a population model Q that satisfies $\frac{dQ}{dt} = \frac{1}{500}Q(220 - Q)$. Find the value of $\frac{dQ}{dt}$ at the time when Q grows most rapidly.

(c) For the population model Q introduced in part (b), use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $Q(10)$. Show the computations that lead to your answer.

(a) $\frac{dP}{dt} = \frac{1}{4}(220 - P)$

$$\int \frac{dP}{220 - P} = \int \frac{1}{4}\,dt$$

$$-\ln|220 - P| = \frac{1}{4}t + C$$

Because $P(0) = 20$, $P < 220$, so $|220 - P| = 220 - P$.

$$-\ln(220 - 20) = \frac{1}{4}(0) + C \Rightarrow C = -\ln 200$$

$$220 - P = 200e^{-t/4}$$

$$P = 220 - 200e^{-t/4},\ t \ge 0$$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } P \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(b) Q satisfies a logistic differential equation with carrying capacity 220. Q grows most rapidly when $Q = \frac{220}{2} = 110$.

$$\left.\frac{dQ}{dt}\right|_{Q=110} = \frac{110^2}{500} = \frac{121}{5}$$

$2 : \begin{cases} 1 : Q = 110 \\ 1 : \text{answer} \end{cases}$

(c) $Q(0) = 20$

$$Q'(0) = \frac{1}{500}(20)(200) = 8$$

$$Q(5) \approx 20 + (8)(5) = 60$$

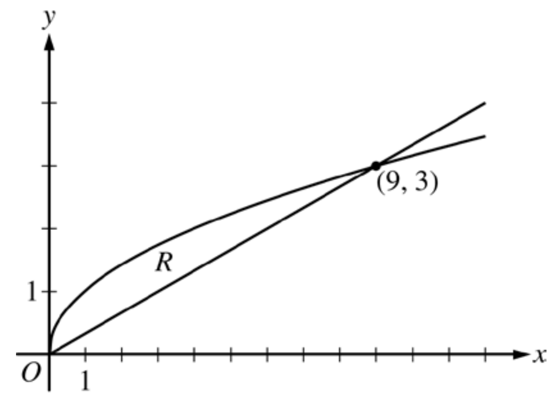
$$Q'(5) \approx \frac{1}{500}(60)(220 - 60) = \frac{96}{5}$$

$$Q(10) \approx 60 + \left(\frac{96}{5}\right)(5) = 156$$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

Question 5

Let R be the region in the first quadrant enclosed by the graphs of $g(x) = \sqrt{x}$ and $h(x) = \frac{x}{3}$, as shown in the figure above.



(a) Find the area of region R.

(b) Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when R is revolved about the horizontal line $y = 4$.

(c) Find the maximum vertical distance between the graph of g and the graph of h between $x = 0$ and $x = 16$. Justify your answer.

(a) Area $= \int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{6}x^2\right]_0^9$

$= \frac{2}{3}\cdot 27 - \frac{1}{6}\cdot 81 = \frac{9}{2}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi\int_0^9 \left[\left(4 - \frac{x}{3}\right)^2 - \left(4 - \sqrt{x}\right)^2\right] dx$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

(c) Consider the function $D(x) = \sqrt{x} - \frac{x}{3}$.

$D'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{3} = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$D'(x) = 0 \Rightarrow x = \frac{9}{4}$

x	$D(x)$	Distance between graphs
0	0	0
$\frac{9}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
16	$-\frac{4}{3}$	$\frac{4}{3}$

The maximum vertical distance between the graph of g and the graph of h between $x = 0$ and $x = 16$ is $\frac{4}{3}$.

$3 : \begin{cases} 1 : \text{sets } D'(x) = 0 \\ 1 : \text{identifies } x = \frac{9}{4} \text{ as a candidate} \\ 1 : \text{answer and justification} \end{cases}$

Question 6

The Maclaurin series for a function f is given by $\frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \cdots + \frac{x^n}{n+2} + \cdots$.

(a) Use the ratio test to find the interval of convergence of the Maclaurin series for f.

(b) Let g be the function given by $g(x) = f(-2x)$. Find the first three terms and the general term of the Maclaurin series for g.

(c) The first two terms of the Maclaurin series for f are used to approximate $f(0.1)$. Given that $|f'''(x)| \le 2$ for $0 \le x \le 0.1$, use the Lagrange error bound to show that this approximation differs from $f(0.1)$ by at most $\frac{1}{3000}$.

(a) Let a_n be the nth term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{n+3} \cdot \frac{n+2}{x^n} = \left(\frac{n+2}{n+3}\right) \cdot x$$

$$\lim_{n \to \infty} \left| \left(\frac{n+2}{n+3}\right) \cdot x \right| = |x|$$

$|x| < 1 \Rightarrow -1 < x < 1$
The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots$.
This series converges by the alternating series test.

When $x = 1$, the series is $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$.
This series diverges since it is a harmonic series.

Therefore, the interval of convergence is $-1 \le x < 1$.

$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

(b) The first three terms are

$$\frac{-2x}{3} + \frac{(-2x)^2}{4} + \frac{(-2x)^3}{5} = -\frac{2}{3}x + x^2 - \frac{8}{5}x^3.$$

The general term is $\frac{(-2x)^n}{n+2}$.

$2 : \begin{cases} 1 : \text{first three terms} \\ 1 : \text{general term} \end{cases}$

(c) $$\text{Error} \le \frac{\max\limits_{0 \le x \le 0.1} |f'''(x)|}{3!} \cdot \left(\frac{1}{10}\right)^3 \le \frac{2}{6} \cdot \frac{1}{1000} = \frac{1}{3000}$$

$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$