Free-Response Scoring Guidelines

The following contains the scoring guidelines for the free-response questions in this exam.

Question 1

The curve C in the xy-plane is given parametrically by $(x(t), y(t))$, where $\frac{dx}{dt} = t\sin\left(\frac{2t}{3}\right)$ and $\frac{dy}{dt} = \cos\left(\frac{t^2}{4}\right)$ for $1 \le t \le 6$.

(a) Find the slope of the line tangent to the curve C at the point where $t = 3$.

(b) For $1 \le t \le 6$, what is the value of t at which the line tangent to the curve C is vertical?

(c) Find the length of the curve C for $1 \le t \le 6$.

(d) Given $y(1) = 2$, find $y(3)$.

(a) Slope $= \frac{y'(3)}{x'(3)} = \frac{\cos(9/4)}{3\sin(2)} = -0.230$	$2 : \begin{cases} 1 : \text{expression for the slope} \\ 1 : \text{answer} \end{cases}$
(b) $\frac{dx}{dt} = t\sin\left(\frac{2t}{3}\right) = 0 \Rightarrow t = \frac{3\pi}{2} \approx 4.712$	$2 : \begin{cases} 1 : \text{sets } \frac{dx}{dt} = 0 \\ 1 : \text{answer} \end{cases}$
(c) Length $= \int_1^6 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 10.569$ (or 10.568)	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) $y(3) = 2 + \int_1^3 y'(t)\, dt = 2.805$ (or 2.804)	$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{initial condition} \\ 1 : \text{answer} \end{cases}$

Question 2

t (weeks)	0	3	6	10	12
$G(t)$ (games per week)	160	450	900	2100	2400

A store tracks the sales of one of its popular board games over a 12-week period. The rate at which games are being sold is modeled by the differentiable function G, where $G(t)$ is measured in games per week and t is measured in weeks for $0 \le t \le 12$. Values of $G(t)$ are given in the table above for selected values of t.

(a) Approximate the value of $G'(8)$ using the data in the table. Show the computations that lead to your answer.

(b) Approximate the value of $\int_0^{12} G(t)\,dt$ using a right Riemann sum with the four subintervals indicated by the table. Explain the meaning of $\int_0^{12} G(t)\,dt$ in the context of this problem.

(c) One salesperson believes that, starting with 2400 games per week at time $t = 12$, the rate at which games will be sold will increase at a constant rate of 100 games per week per week. Based on this model, how many total games will be sold in the 8 weeks between time $t = 12$ and $t = 20$?

(d) Another salesperson believes the best model for the rate at which games will be sold in the 8 weeks between time $t = 12$ and $t = 20$ is $M(t) = 2400e^{-0.01(t-12)^2}$ games per week. Based on this model, how many total games, to the nearest whole number, will be sold during this period?

(a) $G'(8) \approx \dfrac{G(10) - G(6)}{10 - 6} = \dfrac{2100 - 900}{10 - 6} = 300$ games per week per week

1 : approximation

(b) $$\int_0^{12} G(t)\,dt \approx 3 \cdot G(3) + 3 \cdot G(6) + 4 \cdot G(10) + 2 \cdot G(12)$$
$$= 3 \cdot 450 + 3 \cdot 900 + 4 \cdot 2100 + 2 \cdot 2400$$
$$= 17250$$

$\int_0^{12} G(t)\,dt$ represents the total number of games sold over the 12-week period $0 \le t \le 12$.

$3 : \begin{cases} 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \\ 1 : \text{explanation} \end{cases}$

(c) The rate of sales of the game for $12 \le t \le 20$ is $R(t) = 2400 + 100(t - 12)$.

Based on this model, the total number of games that will be sold between time $t = 12$ and $t = 20$ is $\int_{12}^{20} R(t)\,dt = 22400$.

$3 : \begin{cases} 1 : \text{rate function} \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

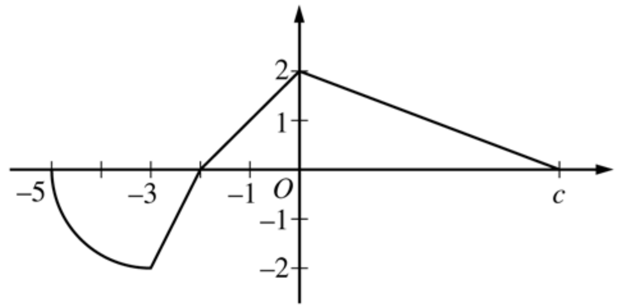
(d) $\int_{12}^{20} M(t)\,dt = 15784.07655$

Based on this model, the total number of games that will be sold between time $t = 12$ and $t = 20$ is 15784.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Question 3

The function f is defined on the interval $-5 \le x \le c$, where $c > 0$ and $f(c) = 0$. The graph of f, which consists of three line segments and a quarter of a circle with center $(-3, 0)$ and radius 2, is shown in the figure above.



Graph of f

(a) Find the average rate of change of f over the interval $[-5, 0]$. Show the computations that lead to your answer.

(b) For $-5 \le x \le c$, let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the x-coordinate of each point of inflection of the graph of g. Justify your answer.

(c) Find the value of c for which the average value of f over the interval $-5 \le x \le c$ is $\frac{1}{2}$.

(d) Assume $c > 3$. The function h is defined by $h(x) = f\left(\frac{x}{2}\right)$. Find $h'(6)$ in terms of c.

(a) The average rate of change of f over the interval $[-5, 0]$ is

$$\frac{f(0) - f(-5)}{0 - (-5)} = \frac{2}{5}.$$

1 : answer

(b) $g'(x) = f(x)$

The graph of g has a point of inflection at $x = -3$ because $g' = f$ changes from decreasing to increasing at this point.

The graph of g has a point of inflection at $x = 0$ because $g' = f$ changes from increasing to decreasing at this point.

$$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -3 \text{ and } x = 0 \\ 1 : \text{justification} \end{cases}$$

(c) $\frac{1}{c+5}\int_{-5}^{c} f(x)\,dx = \frac{1}{2}$

$\frac{1}{c+5}(-\pi + (-1) + 2 + c) = \frac{1}{2}$

$c = 3 + 2\pi$

$$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$$

(d) $h'(x) = \frac{1}{2} f'\left(\frac{x}{2}\right)$

$h'(6) = \frac{1}{2} f'(3) = \frac{1}{2} \cdot \frac{-2}{c} = -\frac{1}{c}$

$$2 : \begin{cases} 1 : h'(x) \\ 1 : \text{answer} \end{cases}$$

Question 4

Let R be the region bounded above by the graph of $y = e^{-x}$, below by the x-axis, on the left by the y-axis, and on the right by the vertical line $x = m$, where $m > 0$ is a constant.

(a) Find the area of R in terms of m.

(b) Region R is revolved around the horizontal line $y = -3$ to form a solid. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(c) Find the value of the real number $k > 1$ such that $\int_0^\infty k^{-x}\,dx = 2$, or show that no such k exists.

(a) Area $= \int_0^m e^{-x}\,dx = -\left[e^{-x}\right]_0^m = -e^{-m} + 1$	$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \pi \int_0^m \left[\left(e^{-x} + 3\right)^2 - 3^2\right] dx$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$
(c) $\lim\limits_{b\to\infty} \int_0^b k^{-x}\,dx = -\frac{1}{\ln k} \lim\limits_{b\to\infty}\left[k^{-x}\right]_0^b = -\frac{1}{\ln k}\lim\limits_{b\to\infty}\left(\frac{1}{k^b} - 1\right) = \frac{1}{\ln k}$ $\frac{1}{\ln k} = 2 \Rightarrow k = e^{1/2}$	$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit} \\ 1 : \text{answer} \end{cases}$

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Question 5

For $0 \le t \le 24$ hours, the temperature inside a refrigerator in a kitchen is given by the function W that satisfies the differential equation $\frac{dW}{dt} = \frac{3\cos t}{2W}$. $W(t)$ is measured in degrees Celsius $(°\text{C})$, and t is measured in hours. At time $t = 0$ hours, the temperature inside the refrigerator is $3°\text{C}$.

(a) Write an equation for the line tangent to the graph of $y = W(t)$ at the point where $t = 0$. Use the equation to approximate the temperature inside the refrigerator at $t = 0.4$ hour.

(b) Find $y = W(t)$, the particular solution to the differential equation with initial condition $W(0) = 3$.

(c) The temperature in the kitchen remains constant at $20°\text{C}$ for $0 \le t \le 24$. The cost of operating the refrigerator accumulates at the rate of \$0.001 per hour for each degree that the temperature in the kitchen exceeds the temperature inside the refrigerator. Write, but do not evaluate, an expression involving an integral that can be used to find the cost of operating the refrigerator for the 24-hour interval.

(a) $\left.\frac{dW}{dt}\right|_{(t,\,W)=(0,\,3)} = \frac{3\cos 0}{2(3)} = \frac{1}{2}$

An equation for the tangent line is $y = \frac{1}{2}t + 3$.

$W(0.4) \approx \frac{1}{2}(0.4) + 3 = 3.2°\text{C}$

$2: \begin{cases} 1: \text{tangent line equation} \\ 1: \text{approximation} \end{cases}$

(b) $2W\ dW = 3\cos t\ dt$

$\int 2W\ dW = \int 3\cos t\ dt$

$W^2 = 3\sin t + C$

$3^2 = 3\sin 0 + C \Rightarrow C = 9$

$W^2 = 3\sin t + 9$

Since $W(0) = 3$, $W = \sqrt{3\sin t + 9}$ for $0 \le t \le 24$.

$5: \begin{cases} 1: \text{separation of variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ \quad \text{and uses initial condition} \\ 1: \text{solves for } W \end{cases}$

Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(c) $0.001\int_0^{24} (20 - W(t))\ dt$ dollars

$2: \begin{cases} 1: \text{integrand} \\ 1: \text{limits and constant} \end{cases}$

Question 6

Let f be a function that has derivatives of all orders for all real numbers, and let $3 - 6(x-1)^2 + 5(x-1)^3 - 2(x-1)^4$ be the fourth-degree Taylor polynomial for the function f about $x = 1$.

(a) Find $f'''(1)$.

(b) Does the graph of f have a relative maximum, a relative minimum, or neither at $x = 1$? Give a reason for your answer.

(c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$. Find the third-degree Taylor polynomial for g about $x = 1$, and use it to approximate $g(1.1)$.

(d) Let h be the function defined by $h(x) = f'(x)$. Find the third-degree Taylor polynomial for h about $x = 1$. Verify that the graph of h does not have a point of inflection at $x = 1$.

Solution	Scoring
(a) $5 = \frac{f'''(1)}{6} \Rightarrow f'''(1) = 30$	1 : answer
(b) $f'(1) = 0$ $-6 = \frac{f''(1)}{2} \Rightarrow f''(1) = -12 < 0$ Therefore, f has a relative maximum at $x = 1$.	$2 : \begin{cases} 1 : f'(1) = 0 \\ 1 : \text{answer with reason} \end{cases}$
(c) Let G_3 be the third-degree Taylor polynomial for g about $x = 1$. $G_3(x) = \int_1^x \left(3 - 6(t-1)^2\right) dt$ $= \left[3(t-1) - 2(t-1)^3\right]_1^x$ $= 3(x-1) - 2(x-1)^3$ $g(1.1) \approx G_3(1.1) = 3(0.1) - 2(0.1)^3 = 0.298$	$3 : \begin{cases} 2 : \text{Taylor polynomial} \\ 1 : \text{approximation} \end{cases}$
(d) Let H_3 be the third-degree Taylor polynomial for h about $x = 1$. $H_3(x) = \frac{d}{dx}\left[3 - 6(x-1)^2 + 5(x-1)^3 - 2(x-1)^4\right]$ $= -12(x-1) + 15(x-1)^2 - 8(x-1)^3$ $15 = \frac{h''(1)}{2} \Rightarrow h''(1) = 30$ Since $h''(1) \neq 0$, the graph of h does not have a point of inflection at $x = 1$.	$3 : \begin{cases} 2 : \text{Taylor polynomial} \\ 1 : \text{reason} \end{cases}$